Contributions to the theory of the Pitot-tube displacement effect

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SUMMARY

In an earlier paper the uniform shear flow past a sphere was studied, by investigating how vortex lines are deformed by the 'primary flow' (flow in the absence of shear), and deducing the 'secondary' vorticity field (first approximation for small shear). In another paper the image system associated with each element of secondary vorticity was found, whence the Biot–Savart law can be used to determine the secondary flow field by integration. The integration is here carried out for the 'downwash' (secondary flow component perpendicular to the undisturbed flow, down the velocity gradient) on the dividing streamline. Difficulties due to the infinite domain of integration and singularities of the integrand are overcome by selecting variables of integration carefully and using known analytical properties of the secondary vorticity. From the computation of downwash is inferred the first approximation (for small shear A) to the 'displacement' δ (displacement of the dividing streamline, up the velocity gradient, far upstream of the sphere). If U is the upstream flow velocity and a the radius of the sphere, the computed value of $\lim_{A\to 0}(U\delta/Aa^2)$ is 0·9.

Details of the calculation show that the secondary trailing vorticity is not an important contributor to the displacement. The downwash is due almost entirely to vorticity upstream of the sphere (Hall's earlier simplified theory gave good results, e.g., 1·24 instead of 0·9, because it concentrated on the effect of local vorticity in producing downwash); further, this produces displacement principally through its image vorticity.

The relation between theories for a sphere and experimental results on Pitot tubes (beginning with Young & Maas 1936) is discussed. Theoretical evidence on tertiary- and quartary-flow effects is used here in the light of recent work which renders the successive-approximation sequence uniformly valid at infinity. The conclusion is that the theories, taken together, are not inconsistent with the experimental evidence that (i) at values of the 'shear parameter' Aa/U at which the displacement is measurable the ratio δ/a seems to have asymptoted to an approximately constant value, and (ii) displacement is greatly reduced in supersonic flow (Johannesen & Mair 1952) or when 'sharp-lipped' tubes are used (Livesey 1956).

1. Introduction

In this paper the mechanism of the 'Pitot-tube displacement effect' is analysed by further study of the shear flow past a sphere, following on three earlier papers, "The image system of a vortex element in a rigid sphere" (Lighthill 1956 a, to be referred to hereafter as **I**), "Drift" (Lighthill 1956 b, with corrigenda in Lighthill 1957 a, to be referred to collectively as **D**), and "The displacement effect of a sphere in a two-dimensional shear flow" (Hall 1956, to be referred to as **H**).

When a parallel shear flow, such as a boundary layer or wake, is investigated with a Pitot tube (that is, a tube with the open end pointing upstream and the other end closed by a manometer), it is found that the pressure in the tube is greater than the total pressure on the streamline approaching the centre of the tube orifice; it is equal rather to the total pressure on a streamline displaced in the direction of higher velocities, by an amount δ usually known as the 'displacement of the effective centre' of the Pitot tube. For the tubes whose ratio of internal diameter d_i to external diameter d_e is 0·6, Young & Maas (1936) found that δ/d_e was scattered fairly randomly about 0·18, and Macmillan (1956) obtained an average value 0·15 for δ/d_e.

Two recently observed phenomena enhance the desirability of understanding the displacement effect physically. First, in supersonic flow, Johannesen & Mair (1952) showed that the displacement effect practically vanished in a wake for $M = 1{\cdot}96$. Understanding of this result is made vastly harder by the presence of the 'bow shock wave' in front of the Pitot tube, but in view of the considerable region of subsonic flow between that shock wave and the orifice, it is safe to say that no explanation can reasonably be attempted until the incompressible-flow phenomenon is understood.

Secondly, although Young & Maas (1936) found some indications that δ/d_e increased slightly with increase of d_i/d_e, Livesey (1956) has made experiments which indicate that there is practically no displacement when $d_i = d_e$. This condition was obtained by chamfering the front part of the tube wall to a cone-frustum shape, so that the orifice became sharp-edged. As a check on his experimental technique, Livesey obtained results for $d_i/d_e = 0{\cdot}6$ comparable with those noted above. The contrasted results are difficult to understand without a rather full discussion of the mechanism of the displacement effect.

It might be thought that useful data for such a discussion would be found by a study of the analogous two-dimensional problem, in which a two-dimensional 'Pitot channel', whose external thickness becomes $2c$ downstream of the nose, is placed in a parallel shear flow. However Hall shows (**H**, Appendix) that the displacement of the dividing streamline in this problem is given by

$$\frac{\delta}{c} = \left(\frac{U^2}{A^2c^2} + 1\right)^{1/2} - \frac{U}{Ac},$$

where $U+Ay$ is the velocity distribution in the oncoming stream*. For values of U/Ac occurring in practice, this result is approximately

$$\frac{\delta}{d_e} = \frac{\delta}{2c} = \frac{Ac}{4U} = \frac{Ad_e}{8U},$$

a value well below the experimentally determined values of the displacement for the values $Ad_e/U < \frac{1}{4}$ which are used in practice. Similar results are found for two-dimensional shear flow about cylinders of various shapes (see Mitchell & Murray 1955 as well as references in **H**). Hall shows, however, that a three-dimensional theory, which takes into account the stretching of the vortex lines of the oncoming shear flow as they pass over the body, but still neglects viscous effects, can be used to obtain displacements of the right order of magnitude. There is a simplifying approximation in Hall's work, which will be critically examined below by comparison with a theory which avoids the approximation, but the results of this examination will be found to be favourable.

Before further detailed discussion it may be worth indicating where the subject stands within hydrodynamics as a whole. It falls under the general heading 'secondary flow' which is used to describe what happens when a parallel shear flow is disturbed in a three-dimensional manner (for example by being led round a bend in a pipe or channel; or deflected by a pressure gradient transverse to the streamlines; or subjected to Coriolis force as in meteorology; or confronted with an obstacle as in the present problem). More particularly, it is used to denote the departure of the disturbed flow from a so-called 'primary flow', in which the streamlines are the same as if the original parallel flow had been uniform instead of sheared.

There are two useful ways of looking at secondary flows. In the older approach (see, for example, Goldstein 1938, chap. 2) one considers what unbalanced pressure gradients are implied by the primary flow when account is taken of inertial forces, altered in magnitude because the original parallel flow is not really uniform. The secondary flow is regarded as maintained by such pressure gradients, and limited by viscous resistance. This view of the matter, though always of qualitative value, leads to useful quantitative results only at rather low Reynolds numbers, when the inertial forces may be regarded as a small perturbation of a régime dominated by viscous resistance (Dean 1927, 1928; Cuming 1951).

In flows at high Reynolds numbers it has therefore become usual (Squire & Winter 1951; Hawthorne 1951, 1954; Hawthorne & Martin 1955; Preston 1954) to seek quantitative results by another method, namely study of the vorticity field. The technique used can be interpreted

* In the last sentence of this Appendix, note that "on the body" should read as "at infinity". (To make $\psi = 0$ the dividing streamline, ψ_1 must tend to $-\frac{1}{2}Ac^2$ as the external body surface becomes flat. But a bounded harmonic function must tend to the same limit at infinity in all directions. Hence also upstream $\psi_1 \to -\frac{1}{2}Ac^2$, so that $\psi \sim Uy+\frac{1}{2}Ay^2-\frac{1}{2}Ac^2$ and the displacement δ of the dividing streamline $\psi = 0$ satisfies (A.9).) Also, in equation (A.8), $\frac{1}{2}A$ should read as $\frac{1}{2}Ay^2$.

geometrically as a study of how vortex elements in the original shear flow are stretched and rotated by the primary flow, yielding a 'secondary vorticity field', whose associated velocity field is the 'secondary flow'. In **D**, a mathematical approach was described which, while essentially equivalent to those of the authors cited, corresponds rather more directly to this geometrical picture.

Such a method is obviously of only approximate validity, partly because it neglects diffusion of vorticity by viscosity or turbulence, and partly because the vortex elements are in reality stretched not by the hypothetical primary flow but by the exact flow. The latter inaccuracy can in principle be reduced by calculating a 'tertiary vorticity field' by considering the stretching and rotation of vortex lines by the combined primary and secondary flows, and deducing a 'tertiary flow' from it, and one can even consider quartary flows, as Hall was able to do (**H**, p. 154) by the use of his simplifying approximation.

The vortex approach to secondary flow has so far had the greatest success with internal flows (through bends in pipes, or cascades), for which the calculation of the velocity field from the vorticity field can be fairly straightforward. With external secondary flows, such as those resulting from parallel shear flow past an obstacle, this calculation is more difficult, and work apart from Hall's (**H**) has been limited to calculations of the secondary vorticity field (Hawthorne 1954; Hawthorne & Martin 1955; and **D**, §2 and §7). The problem of how to deduce the secondary velocity field without making Hall's approximation is considered in some detail in this paper, partly to establish the value of Hall's approximation, partly to improve understanding of the effect, and partly to improve techniques for the mathematical calculation of three-dimensional, fully rotational flows.

The calculations are still confined to the case of a sphere, which can be regarded in many ways as a typical bluff obstacle. The work was consciously prepared for in **D** by the calculation of the secondary vorticity field, and in **I** by the calculation of the image system of a single vortex element in the sphere. This type of image approach, combined with the Biot–Savart law, is found to be more convenient in the present problem than any approach based on images of complete vortex lines.

The relevance of calculations for a sphere to the Pitot-tube problem is supported by the finding that vorticity downstream of the sphere contributes negligibly to the displacement effect. This also kills a plausible hypothesis, namely that secondary trailing vorticity might be largely responsible for the displacement effect, which therefore would be greatly reduced at supersonic speeds for which the trailing vorticity has no upstream influence.

A more correct physical interpretation of the results, as discussed in §4, is that in all cases the image system of the vorticity upstream of the obstacle is the main cause of the displacement. It is shown in §5 how this can be regarded as responsible for the altered situation when either a supersonic main stream or a sharp-lipped orifice is used.

2. Evaluation of the secondary downwash ahead of the sphere

The shear flow past a sphere will here be studied in the notation set out in **D**. Thus, Cartesian axes are used, with origin at the centre of the sphere, such that far upstream the velocity field is

$$v_x = U + Ay, \qquad v_y = v_z = 0, \tag{1}$$

where suffixes denote components. In addition, spherical polar coordinates r, θ, λ such that

$$x = r\cos\theta, \qquad y = r\sin\theta\cos\lambda, \qquad z = r\sin\theta\sin\lambda \tag{2}$$

are used; in these coordinates the sphere is $r = a$.

The primary flow is the irrotational flow about the sphere, associated with an upstream velocity field as in (1) but with $A = 0$. The secondary flow represents the first-order correction for non-zero A: thus, it is the term in Aa/U in an expansion in powers of that non-dimensional parameter.

The secondary vorticity field was determined in **D**, § 7. Certain asymptotic properties of the secondary velocity field were derived in **D**, § 3. Here, such further properties of it are derived as are necessary to determine the displacement of the stagnation streamline to the first order in the parameter Aa/U.

For this one must know the distribution along the line $y = z = 0$, $x < -a$ (which *is* the stagnation streamline for $A = 0$) of the secondary flow component normal to that line. This component is actually in the negative y-direction. It may be referred to as 'downwash' if we think of the upstream velocity distribution as horizontal and increasing upwards. Accordingly, we write $D(s)$ for the value of $(-v_y)$ at the point $(-s, 0, 0)$, and call $D(s)$ the 'downwash function'.

Now, the secondary flow may be calculated in three parts as explained in **D**, § 3. First, there is a part, which is the gradient of a potential ϕ_1 vanishing at infinity, and whose normal velocity component on the surface just cancels out that associated with the uniform shearing motion $v_x = Ay$, $v_y = v_z = 0$. In the case of a sphere this condition may be satisfied by a single solid harmonic (easily obtained by inspection), namely

$$\phi_1 = \frac{Aa^5xy}{3r^5}. \tag{3}$$

This part may be regarded as the velocity field due to the image system of the undisturbed vorticity field $(0, 0, -A)$ in the sphere $r = a$; we see that this image system consists of a single quadrupole at the centre. The contribution of this part to the downwash function $D(s)$ is

$$D_1(s) = \frac{Aa^5}{3s^4}. \tag{4}$$

Secondly, there is the Biot–Savart field of the 'vorticity change' $\boldsymbol{\omega}_1$, that is, the difference between the secondary vorticity field and its undisturbed value $(0, 0, -A)$. The contribution of this Biot–Savart field (**D**, (82)) to $D(s)$ is

$$D_2(s) = -\frac{1}{4\pi}\iiint_{x^2+y^2+z^2>a^2} \frac{\omega_{1x}z - \omega_{1z}(x+s)}{\{(x+s)^2+y^2+z^2\}^{3/2}}\,dxdydz. \tag{5}$$

To evaluate (5) it is convenient to change over to spherical polar coordinates, when it becomes

$$D_2(s) = \frac{1}{4\pi}\int_a^\infty\int_0^\pi\int_0^{2\pi} \times$$
$$\times\left\{\frac{\omega_{1r}\, s\sin\theta\sin\lambda + \omega_{1\theta}(s\cos\theta + r)\sin\lambda + \omega_{1\lambda}(s + r\cos\theta)\cos\lambda}{(s^2+2sr\cos\theta+r^2)^{3/2}}\right\}\times$$
$$\times\, r^2\sin\theta\, dr d\theta d\lambda. \quad (6)$$

Thirdly, there is a further irrotational part of the secondary velocity field, which must be added to the Biot–Savart field of $\boldsymbol{\omega}_1$ so that together they may satisfy the boundary condition on the surface of the sphere. In **I** this was shown to be simply the Biot–Savart field of a certain system of image vorticity inside the sphere. Using the term 'strength' to denote the product of the volume dV of an elementary region with the vorticity inside it, it was shown that transverse vortex elements of strengths $\omega_\theta dV$ and $\omega_\lambda dV$ have images of strengths $(-a/r)\omega_\theta dV$ and $(-a/r)\omega_\lambda dV$ respectively at the inverse point $(a^2/r, \theta, \lambda)$ while a radial vortex element of strength $\omega_r dV$ has an image system consisting of (i) a radial vortex element of strength $(+a/r)\omega_r dV$ at the inverse point, and (ii) a uniform line vortex, of strength $(-1/a)\omega_r dV$ per unit length, stretching between the inverse point and the origin.

It follows by comparison with (6) that the contribution of this image vorticity to the downwash function $D(s)$ is

$$D_3(s) = \frac{1}{4\pi}\int_a^\infty\int_0^\pi\int_0^{2\pi}\left[\left(\frac{a}{r}\right)\times\right.$$
$$\times\frac{\omega_{1r}\, s\sin\theta\sin\lambda - \omega_{1\theta}(s\cos\theta + a^2/r)\sin\lambda - \omega_{1\lambda}\{s + (a^2/r)\cos\theta\}\cos\lambda}{\{s^2+2s(a^2/r)\cos\theta+(a^2/r)^2\}^{3/2}} -$$
$$\left. -\frac{\omega_{1r}}{a}\,\frac{\sin\lambda}{s\sin\theta}\left\{\frac{s\cos\theta + a^2/r}{\{s^2+2s(a^2/r)\cos\theta+(a^2/r)^2\}^{1/2}} - \cos\theta\right\}\right] r^2\sin\theta\, dr d\theta d\lambda, \quad (7)$$

where the coefficient of $(-\omega_{1r}/a)$ inside the square brackets is obtained by integrating with respect to r from 0 to a^2/r (the value of r at the inverse point) the coefficient of ω_{1r} in the curly brackets in (6).

Now, $\omega_{1\lambda}$ takes the simple form (**D**, (81))

$$\omega_{1\lambda} = (A\cos\lambda)\left\{1 - \left(1 - \frac{a^3}{r^3}\right)^{-1/2}\right\}. \quad (8)$$

As a result, the terms involving $\omega_{1\lambda}$ in $D_2(s) + D_3(s)$, which may be designated as $D_\lambda(s)$, can be greatly simplified. Since

$$\int_0^\pi \frac{(s + r\cos\theta)\sin\theta\, d\theta}{(s^2+2sr\cos\theta+r^2)^{3/2}} = \begin{cases} 2s^{-2} & (s > r) \\ 0 & (s < r) \end{cases}, \quad (9)$$

we have

$$D_\lambda(s) = \frac{1}{2s^2}\left[\int_a^s\left\{1 - \left(1 - \frac{a^3}{r^3}\right)^{-1/2}\right\} r^2\, dr - \int_a^\infty\left\{1 - \left(1 - \frac{a^5}{r^3}\right)^{-1/2}\right\} ar\, dr\right]$$
$$= \frac{1}{2s^2}\left[\frac{1}{3}\left\{s^3 - a^3 - s^{3/2}(s^3 - a^3)^{1/2} - \cosh^{-1}\left(\frac{s}{a}\right)^{3/2}\right\} + 0{\cdot}850a^3\right], \quad (10)$$

the last term (independent of s) being obtained by evaluating the infinite integral in terms of factorial functions.

The vorticity components ω_{1r} and $\omega_{1\theta}$ can be written

$$\omega_{1r} = (A\sin\lambda)F_r(r,\theta), \qquad \omega_{1\theta} = (A\sin\lambda)F_\theta(r,\theta), \tag{11}$$

where in **D**, § 7 the quantities F_r and F_θ were tabulated for points on three particular streamlines, and determined asymptotically near the axis of symmetry, near the surface of the sphere and far from the sphere. In terms of F_r and F_θ the terms involving ω_{1r} and $\omega_{1\theta}$ in $D_2(s)+D_3(s)$, which may be designated as $D_r(s)$ and $D_\theta(s)$ respectively, may be written as follows:

$$D_r(s) = \tfrac{1}{4}A\int_a^\infty\int_0^\pi F_r(r,\theta)r^2\sin\theta\,dr\,d\theta\Bigg[\frac{s\sin\theta}{(s^2+2sr\cos\theta+r^2)^{3/2}} + \left(\frac{a}{r}\right)\frac{s\sin\theta}{\{s^2+2s(a^2/r)\cos\theta+a^4/r^2\}^{3/2}} - \frac{1}{as\sin\theta}\times \left\{\frac{s\cos\theta+a^2/r}{\{s^2+2s(a^2/r)\cos\theta+a^4/r^2\}^{1/2}}-\cos\theta\right\}\Bigg], \tag{12}$$

$$D_\theta(s) = \tfrac{1}{4}A\int_a^\infty\int_0^\pi F_\theta(r,\theta)r^2\sin\theta\,dr\,d\theta\Bigg[\frac{s\cos\theta+r}{(s^2+2sr\cos\theta+r^2)^{3/2}} - \left(\frac{a}{r}\right)\frac{s\cos\theta+a^2/r}{\{s^2+2s(a^2/r)\cos\theta+a^4/r^2\}^{3/2}}\Bigg]. \tag{13}$$

In the numerical evaluation of the integrals (12) and (13) difficulties arise from two main sources. First, there are the ordinary difficulties involved in the numerical evaluation of any integral when the domain of integration is infinite. These (as we shall see) can be overcome in the ordinary manner, because of our rather full knowledge of the behaviour of the integrands for large r/a.

The second source of difficulties is the unboundedness of the integrands. For every s there is one value of r (namely $r = s$) for which one of the denominators in (12) and (13) vanishes on $\theta = \pi$. In addition, F_r and F_θ have singularities, both on $r = a$, where F_θ is actually infinite, and on $\theta = 0$, where F_r is actually infinite. Thus, there are difficulties in the neighbourhood of the whole dividing streamline ($\theta = \pi$, $r = a$, $\theta = 0$).

This indicates the advantage of a change in the variables of integration. Instead of r, θ we use ρ_0, θ, where ρ_0, introduced in **D**, § 5 and defined by the equation (**D**, (60))

$$r^2\left(1-\frac{a^3}{r^3}\right)\sin^2\theta = \rho_0^2, \tag{14}$$

is constant along each streamline*. Then the integration with respect to θ can be carried out first (and here the fact that this integration is over a finite range $0 < \theta < \pi$ is an advantage); the singular character of the integrands comes in only when the second integration (with respect to ρ_0)

* Stokes's stream function $\frac{1}{2}U\rho_0^2$ would do almost as well as ρ_0 as a variable of integration, but there are some small advantages in favour of ρ_0 in the present problem.

is carried out, and there it is confined to the single point $\rho_0 = 0$ which represents the whole of the dividing streamline.

To change the integrals (12) and (13) into integrals with respect to ρ_0 and θ, one has only to replace the differential product $drd\theta$ in each by

$$drd\theta = \left(\frac{\partial r}{\partial \rho_0}\right)_{\theta\,\text{constant}} d\rho_0\, d\theta = \frac{r(r^3 - a^3)}{\rho_0(r^3 + \frac{1}{2}a^3)} d\rho_0\, d\theta. \tag{15}$$

The ranges of integration become $0 < \theta < \pi$, $0 < \rho_0 < \infty$.

The integrations with respect to θ, keeping ρ_0 constant, were carried out by Simpson's rule, with equal intervals of 10° for θ, for three values (0·25, 0·5 and 1·0) of ρ_0/a, using the values of r/a, F_r and F_θ given in **D**, table 3, for the required combinations of values of θ and ρ_0/a. At a few places, where the values of the integrands made it appear probable that Simpson's rule was being slightly strained, intermediate values were computed and used to improve the approximation.

If the result of such integration is written as $D_r(s, \rho_0)$ in the case of the integrand of $D_r(s)$, so that

$$D_r(s) = \int_0^\infty D_r(s, \rho_0)\, d\rho_0, \tag{16}$$

and similarly with D_θ, the problem remains of carrying out the integration with respect to ρ_0. This was done by using Simpson's rule in different forms for two parts of the integral, writing

$$\begin{aligned}\int_0^\infty D_r(s,\rho_0)\, d\rho_0 &= \int_0^{\frac{1}{2}a} D_r(s,\rho_0)\, d\rho_0 + \int_0^{2/a} D_r(s,\rho_0)\rho_0^2\, d(\rho_1^{-1}) \\ &\doteqdot \frac{1}{12} a[D_r(s,0) + 4D_r(s,\tfrac{1}{4}a) + D_r(s,\tfrac{1}{2}a)] + \\ &\quad + \frac{1}{3a}\left[\tfrac{1}{4}a^2 D_r(s,\tfrac{1}{2}a) + 4a^2 D_r(s,a) + \lim_{\rho_0\to\infty}\{\rho_0^2 D_r(s,\rho_0)\}\right].\end{aligned} \tag{17}$$

In this method one requires only those three values of $D_r(s, \rho_0)$ which were computed as described above, and the two limiting values $D_r(s, 0)$ and $\lim_{\rho_0\to\infty}\{\rho_0^2 D_r(s,\rho_0)\}$. An equation similar to (17) is used also for D_θ.

Now it may be shown that both $D_r(s, 0)$ and $D_\theta(s, 0)$ are zero. This result is deduced from the detailed behaviour of F_r and F_θ as $\rho_0 \to 0$ derived in **D**, § 7 (equations (78) to (80)). The singularity at $\theta = \pi$, $r = s$ due to inverse (3/2)th powers in (12) and (13) produces no effect in $D_r(s, 0)$ because F_r vanishes on $\theta = \pi$, and none in $D_\theta(s, 0)$ because of the factor $s\cos\theta + r$ in the numerator. The singularity of F_θ on $r = a$ is cancelled out by the vanishing on $r = a$ of the term in square brackets in (13). The contribution to $D_\theta(s, 0)$ as well as to $D_r(s, 0)$ of this part of the streamline $\rho_0 = 0$ therefore vanishes, since the coefficient of $d\rho_0\, d\theta$ in the differential element (15) vanishes as $\rho_0 \to 0$ for fixed θ. Finally, the singularity in F_r as $\theta \to 0$ produces no effect in $D_r(s, 0)$ because of the vanishing as $\theta \to 0$ of the term in square brackets in (12).

The limits as $\rho_0 \to \infty$, on the other hand, do not both vanish. By (**D**, (77))

$$F_\theta \sim -\frac{1}{2}\left(\frac{a}{r}\right)^3 \cos\theta, \quad F_r \sim \left(\frac{a}{r}\right)^3 \sin\theta, \quad r \sim \rho_0 \operatorname{cosec}\theta, \tag{18}$$

as $\rho_0 \to \infty$. Hence by (13) and (15)

$$D_\theta(s,\rho_0) \sim \tfrac{1}{4}A \int_0^\pi \left\{-\frac{1}{2}\left(\frac{a}{\rho_0}\right)^3 \sin^3\theta\cos\theta\right\}(\rho_0^2 \operatorname{cosec}^2\theta\, d\theta)\left(-\frac{a}{\rho_0 s^2}\cos\theta\sin\theta\right) = \frac{\pi A a^4}{64\rho_0^2 s^2}, \tag{19}$$

but

$$D_r(s,\rho_0) \sim \tfrac{1}{4}A \int_0^\pi \left\{\left(\frac{a}{\rho_0}\right)^3 \sin^4\theta\right\}(\rho_0^2 \operatorname{cosec}^2\theta\, d\theta)\left\{O\left(\frac{1}{r^2}\right)\right\} = O\left(\frac{1}{\rho_0^3}\right), \tag{20}$$

so that

$$\lim_{\rho_0\to\infty}\{\rho_0^2 D_\theta(s,\rho_0)\} = \frac{\pi A a^4}{64 s^2}, \qquad \lim_{\rho_0\to\infty}\{\rho_0^2 D_r(s,\rho_0)\} = 0. \tag{21}$$

Note that the distant transverse vorticity is effective entirely through its images in the sphere (that is, only the second term in square brackets in (13) contributes to the asymptotic result (19)); the distant radial vorticity is less effective* because its image system consists of a nearly cancelling pair of vortex elements of opposite sense.

Everything has now been found to enable $D_r(s)$ and $D_\theta(s)$ to be determined from equation (17). This has been done for $s/a = 1$ and $\sqrt{2}$, and the results are given in table 1, together with values of D_λ (see (10)) and D_1 (see (4)), the contributions from the 'ring vorticity' $\omega_{1\lambda}$ and from the image system of the undisturbed vorticity distribution. The four terms are added to produce the total downwash given in the last column.

s/a	D_r/Aa	D_θ/Aa	D_λ/Aa	D_1/Aa	Total D/Aa
1	$0{\cdot}12_7$	$0{\cdot}08_9$	0·425	0·333	0·97
$\sqrt{2}$	$0{\cdot}11_6$	$0{\cdot}04_7$	0·083	0·083	0·33

Table 1.

For comparison with these values of $D(s)$ we have for large s/a, by (**D**, (86)),

$$D(s) = (-v_{1y})_{x=-s,y=z=0} \sim \frac{A(V_b + \frac{1}{2}V_h)}{4\pi s^2} = \frac{5}{12}\left(\frac{a}{s}\right)^2 Aa, \tag{22}$$

where we have used the facts that the volume V_b of the sphere is $\frac{4}{3}\pi a^3$, and the volume V_h of fluid, whose mass is the 'hydrodynamic mass' associated with the sphere's motion, is $\frac{1}{2}V_b$. Table 1 shows that $(s/a)^2 D(s)$ is 0·97 for $s/a = 1$ and 0·66 for $s/a = \sqrt{2}$, so that it is tending fairly rapidly to the asymptotic value $\frac{5}{12} = 0{\cdot}42$ as $s/a \to \infty$.

* Actually, a more detailed analysis than that given above shows that

$$D_r(s,\rho_0) = O(\rho_0^{-4}).$$

3. First-order displacement of the stagnation streamline

The stagnation streamline, or 'dividing streamline', must be in the plane of symmetry $z = 0$ (or $\lambda = 0$) and on it

$$v_r\, r\, d\theta - v_\theta\, dr = 0. \tag{23}$$

For the irrotational flow about a sphere this equation becomes

$$U\cos\theta\left(1-\frac{a^3}{r^3}\right)r\,d\theta + U\sin\theta\left(1+\frac{a^3}{2r^3}\right)dr = 0, \tag{24}$$

which can be put into the form

$$d\left\{r\sin\theta\left(1-\frac{a^3}{r^3}\right)^{1/2}\right\} = 0, \tag{25}$$

after multiplying through by $U^{-1}(1-a^3/r^3)^{-1/2}$. The solution of (25) that must be used is

$$r\sin\theta\left(1-\frac{a^3}{r^3}\right)^{1/2} = 0, \tag{26}$$

since other solutions give $r\sin\theta \to \infty$ as $r \to a$, which is impossible if the streamline is to meet the body. Hence $\theta = 0$ or π off the sphere $r = a$.

When $A \neq 0$ we can obtain the displacement of the upstream part $\theta = \pi$ of the stagnation streamline by modifying the v_θ term in (23) by including the downwash function $D(s)$ of § 2 in it. The term in v_r need not have a secondary flow term included, however, since it is multiplied by $r d\theta$ which is itself a small* quantity of order A (vanishing, as we have seen, for $A = 0$).

Thus, (24) is replaced by

$$U\cos\theta\left(1-\frac{a^3}{r^3}\right)r\,d\theta + \left\{U\sin\theta\left(1+\frac{a^3}{2r^3}\right) - D(r)\right\}dr = 0. \tag{27}$$

Equation (27) after multiplication by $U^{-1}(1-a^3/r^3)^{-1/2}$, an integrating factor already used above, becomes

$$d\left\{r\sin\theta\left(1-\frac{a^3}{r^3}\right)^{1/2}\right\} = \frac{D(r)\,dr}{U(1-a^3/r^3)^{1/2}}. \tag{28}$$

Hence the dividing streamline takes the form

$$r\sin\theta = \frac{1}{(1-a^3/r^3)^{1/2}}\int_a^r \frac{D(s)\,ds}{U(1-a^3/s^3)^{1/2}}, \tag{29}$$

that solution of (28) being chosen for which $r\sin\theta$ remains finite as $r \to a$. Thus, the displacement of the stagnation point on the sphere is

$$\lim_{r\to a}(r\sin\theta) = \frac{2a}{3U}D(a) = 0{\cdot}65\,\frac{Aa^2}{U}, \tag{30}$$

* Actually, $r\,d\theta$ becomes less small as $r \to a$, as the solution to be obtained shows, but since $v_r \to 0$ as $r \to a$ the argument is unaffected.

where table 1 has been used, while the displacement δ of the stagnation streamline far upstream of the sphere (the main object of our investigation) is

$$\delta = \lim_{r\to\infty}(r\sin\theta) = \frac{1}{U}\int_a^\infty \frac{D(s)\,ds}{(1-a^3/s^3)^{1/2}}. \tag{31}$$

To compute δ numerically we substitute $s = a\operatorname{cosec}\alpha$ in (31), giving

$$\frac{U\delta}{Aa^2} = \int_0^{\frac{1}{2}\pi} \frac{D(a\operatorname{cosec}\alpha)\operatorname{cosec}\alpha\cot\alpha}{Aa(1-\sin^3\alpha)^{1/2}}\,d\alpha. \tag{32}$$

If Simpson's rule with interval $\frac{1}{4}\pi$ is used in (32), we obtain

$$\frac{U\delta}{Aa^2} \doteqdot \frac{1}{12}\pi\left[\lim_{s\to\infty}\left\{\frac{s^2D(s)}{Aa^3}\right\} + 4\,\frac{D(a\surd 2)}{Aa}\,\frac{\surd 2}{(1-2^{-3/2})^{1/2}} + \frac{D(a)}{Aa}\lim_{\alpha\to\frac{1}{2}\pi}\frac{\cos\alpha}{(1-\sin^3\alpha)^{1/2}}\right] \tag{33}$$

$$= \frac{1}{12}\pi\left\{\frac{5}{12} + 7{\cdot}036\,\frac{D(a\surd 2)}{Aa} + \frac{2}{3}\,\frac{D(a)}{Aa}\right\}$$

$$= \frac{1}{12}\pi(0{\cdot}417 + 2{\cdot}3_{22} + 0{\cdot}6_{47})$$

$$= 0{\cdot}8_9 = 0{\cdot}9. \tag{34}$$

The use of a large interval $\frac{1}{4}\pi$ for Simpson's rule may appear drastic, but the numerical value of the terms, not far from being in the ratio 1:4:1, indicates a smoothness in the integrand which justifies the conclusion that $U\delta/Aa^2 = 0{\cdot}9$ to one place of decimals. The integrations with respect to θ, unless 19 values had been used and a few more interpolated here and there, were likely to have caused more error than those with respect to ρ_0 or s which used 5 and 3 values respectively but had far greater smoothness in the integrands. Greater accuracy than one place of decimals is not required as the theory is only an asymptotic one for small Aa/U in any case.

It is really a quadruple integration which has been performed to get (34). One has integrated with respect to s a function part of which consisted of integrations with respect to ρ_0 and θ of functions which in **D** were derived as derivatives of the 'drift function' t, which is itself defined as an integral. The result of all this work, which it has taken three papers to expound (although the first two, **D** and **I**, each had, in addition, a more general application), is principally the numerical value (34). However, it will be seen in the next section that some useful qualitative results can also be found by studying intermediate steps of the calculation.

4. Discussion of results and comparison with Hall's theory

It is helpful to begin discussing the results of §2 and §3 by comparing them with those of Hall's much simpler theory of the same shear flow past a sphere. This is based on one principal approximate assumption (**H**, p. 145): that on the plane of symmetry $z = 0$ the velocity gradient $\partial v_z/\partial z$ takes the same value as in the primary (irrotational) flow. This assumption determines the velocity field (v_x, v_y) in the plane of symmetry as the sum of the primary velocity field and an additional velocity field satisfying the two-dimensional equation of continuity

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{35}$$

The only non-zero vorticity component

$$\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \tag{36}$$

in this plane can also be determined from Helmholtz's equation

$$v_x \frac{\partial \omega_z}{\partial x} + v_y \frac{\partial \omega_z}{\partial y} = \omega_z \frac{\partial v_z}{\partial z}, \tag{37}$$

since the value which $\partial v_z/\partial z$ takes on the plane $z = 0$ has already been assumed. The two-dimensional velocity field (v_x, v_y) can then be determined without reference to conditions outside that plane, from a knowledge of its divergence and curl, from the boundary condition on the body, and suitable boundary conditions at infinity.

The latter boundary conditions have to be altered somewhat from their original form. Two-dimensional flows about obstacles have the well-known lack of uniqueness arising from arbitrariness in the circulation; Hall avoids this (tacitly, in **H**, (2.15)) by selecting only the solution in which the disturbance velocities fall off as fast as the inverse square of the distance from the origin (as indeed they must in the real flow); other solutions, with additional vorticity inside the sphere, would have disturbance velocities falling off as the inverse first power. On the other hand, to obtain existence of solutions, the condition that disturbance velocities tend to zero as $r \to \infty$ has to be applied only in a bounded interval of y (**H**, (3.44)), but again this is reasonable because the primary flow is a sensible first approximation to the motion only in such a region.

With these assumptions the equations are solved by means of a power series in the non-dimensional shear parameter Aa/U. The coefficient of Aa/U is an approximation to the secondary flow discussed in §2 and §3 above. The coefficients of $(Aa/U)^2$ and $(Aa/U)^3$, also obtained, are approximations to what may be called the tertiary and quartary flows.

In order to form an estimate of the value of Hall's approximate assumptions, we may compare his results for the secondary downwash function $D(s)$ with those obtained in §2 and §3. He gets, in our notation,

$$\frac{D(s)}{Aa} = \frac{a^3}{2s^3} + \frac{a}{s} \int_{s/a}^{\infty} \{x^{5/2}(x^3-1)^{-1/2} - x\}\, dx. \tag{38}$$

To check this, see **H**, equations (2.5), (2.9), (2.11), (3.16), (3.17) and (3.21); in Hall's notation, equation (38) is

$$(-v)_{\theta=0} = UK\left(\frac{1}{2r^3} - \frac{dQ}{dr}\right). \tag{39}$$

Table 2 compares values of the secondary downwash function given by this paper and by Hall's theory (that is, by (38) above). It is seen that Hall's method of approximation does not lead to any large departure from our more exact values. His downwash is an overestimate; so, therefore, is his inferred value of $U\delta/Aa^2$, which, as deduced from (38) and (31), is 1·24. For comparison, if we integrate (31) approximately by means of equation (33), using Hall's values of $D(s)$ given in table 2, we obtain $U\delta/Aa^2 = 1{\cdot}20$. Neither result exceeds our value 0·9, obtained with far greater labour, by as much as 40%.

	$D(a)/Aa$	$D(a\sqrt{2})/Aa$	$\lim_{s\to\infty}(s^2D(s)/Aa^3)$
This paper	0·97	0·33	0·42
H	1·39	0·45	0·50

Table 2.

Now, in discussing the reasons for the success of Hall's approach to the problem of estimating secondary downwash and the displacement effect, it must be remembered that he uses the correct secondary vorticity field in the plane $z = 0$. (For, in calculating ω_z to the first approximation, one may legitimately put $\partial v_z/\partial z$ in (37) equal to its primary-flow value; and indeed Hall's expression for ω_z in the plane $z = 0$ agrees with our equation (8).) His secondary flow field in that plane, however, is simply *one* velocity field which satisfies the boundary condition and has the right vorticity in the plane (to be precise, it is the one whose two-dimensional divergence vanishes). In reality, the secondary velocity field can be determined only from the complete secondary vorticity field, both on the plane and off, as in § 2 above. It is easily checked that the two-dimensional divergence of this field in the plane $z = 0$ will not in general be zero.

Thus, Hall's assumptions are equivalent to the view that the main thing is to pick a velocity field which has the right vorticity locally, on the assumption that the vorticity field near the plane $z = 0$, with its image vorticity, will be much more potent in generating the velocity distribution on $z = 0$ than is the vorticity field away from $z = 0$ (and, in particular, the secondary trailing vorticity) in combination with *its* images. Now, this statement was already verified far upstream of the body in **D**, § 3 and corrigenda, from which it appeared that such an approach would give a value of the secondary downwash only 20% above the exact value far upstream, as is now confirmed by the results in table 2. For it was shown (**D**, (19) and (23)) that the velocity field far upstream which results directly from the local vorticity distribution is asymptotically

$$\frac{A(V_b+V_h)}{4\pi}\left(-\frac{y}{r^3}, \frac{x}{r^3}, 0\right), \tag{40}$$

but that the complete asymptotic form includes also a term associated with the trailing vorticity, which (**D**, (86)) changes the $V_b + V_h$ in the downwash implied by (40) into $V_b + \frac{1}{2}V_h$, as was used in (22) above. Thus in the case of a sphere (for which $V_h = \frac{1}{2}V_b$) expression (40) is an overestimate by exactly 20%, which identifies it with Hall's result in this region.

Nearer the sphere itself Hall's theory still gives reasonably good results for the secondary downwash. To explain this it is desirable to find out how much of the secondary downwash calculated at $s/a = 1$ and $\surd 2$ in § 2 comes from trailing vorticity alongside and downstream of the sphere, which was ignored in Hall's theory.

In addition, there is an independent interest in asking how much contribution comes from trailing vorticity in the region where a turbulent wake would be found in the real flow, because of course the secondary vorticity distribution which has been used would be seriously inaccurate in this region. Accordingly, the contributions to $D(a)$ and $D(a\surd 2)$ from vorticity components ω_{1r} and $\omega_{1\theta}$ on parts of the streamlines $\rho_0/a = 0{\cdot}25$ and $0{\cdot}5$ with $\theta < 90^\circ$, and on parts of the streamline $\rho_0/a = 1$ with $\theta < 45^\circ$, were separately evaluated. Their contribution to $U\delta/Aa^2$ was found to be exclusively negative, but amounted only to $-0{\cdot}06$ altogether. There is some indication here that the applicability of the solution will not be much affected by the inaccuracy of the assumed distribution of vorticity in the wake.

The remaining contributions to $U\delta/Aa^2$ from ω_{1r} and $\omega_{1\theta}$ are all positive; examination shows, however, that the principal contributions come from fairly near the upstream axis $\theta = \pi$. For example, positive contributions from the region $\theta < 120^\circ$ amount to $+0{\cdot}06$, just cancelling the (also small) negative contribution from the wake region.

It is concluded that trailing vorticity cannot be regarded as responsible for the downwash on the axis upstream of the sphere, and this is compatible with the good accuracy of Hall's approximation. Physically, the result is due mainly to cancellation of the effect of trailing vortices by the effect of their image vortices.

Having examined the detailed computations to dispose of the suggestion that trailing vorticity is responsible for the displacement effect, we may now ask what, physically speaking, *is* principally responsible for it. Again, the computation in § 2 and § 3 appears to give a clear answer, namely: of the remaining vorticity and its images, the images make the main contribution to the downwash function $D(s)$ and so to the displacement effect.

To show this, one repeats the calculation of § 2 and § 3 with the terms due to ω_{1r} and $\omega_{1\theta}$ for $\theta < 120^\circ$ omitted (these terms were described above as due to trailing vorticity, and their net contribution to $U\delta/Aa^2$ was found to be zero), and divides the remaining terms into those due to vorticity outside the sphere and those due to image vorticity. Then the contribution to $U\delta/Aa^2$ of the former is found to be $0{\cdot}08$ and the contribution of the latter $0{\cdot}81$.

Actually, the vorticity outside the sphere makes its contribution almost wholly in the region far upstream; in fact the result $\lim_{s\to\infty}\{s^2 D(s)/Aa^2\} = \frac{5}{12}$ is entirely due to vorticity outside the sphere, and this contributes $0{\cdot}11$ to

$U\delta/Aa^2$ in the calculation. On the other hand, downwash near the sphere is almost entirely due to image vorticity; the downwash at $s/a = 1$ due to vorticity outside the sphere is only $0{\cdot}15Aa$ (compare $0{\cdot}82Aa$ due to image vorticity), and that at $s/a = \sqrt{2}$ is $-0{\cdot}03Aa$ (compare $0{\cdot}36Aa$ due to image vorticity); their combined contribution to $U\delta/Aa^2$ is $-0{\cdot}03$.

To sum up the results of this section, the expected effect of trailing vorticity (or more precisely of all the vorticity ω_{1r} and $\omega_{1\theta}$ for $\theta < 120^\circ$) is cancelled out by that of its image vorticity; the effect of the remaining vorticity outside the sphere, on the other hand, is small compared with that of its image vorticity.

The images of the undisturbed vorticity distribution $(0, 0, -A)$, and of the 'ring-vortex' part $\omega_{1\lambda}$ of the change from the undisturbed distribution, are especially potent in producing downwash, as is obvious geometrically from the rule for constructing image vortices, and they account for a part 0·68 in $U\delta/Aa^2$. The images of the vortex elements ω_{1r} and $\omega_{1\theta}$ in $\theta > 120^\circ$ by contrast account for only 0·13.

5. The Pitot-tube problem discussed in the light of the results for a sphere

Now, calculations like those of this paper are difficult to apply to the problem of the Pitot-tube displacement effect, both because of the substantial difference of shape between a sphere and a Pitot tube, and because the calculated amount of the secondary flow, and hence also of the displacement, increases linearly with the shear parameter Aa/U; by contrast, in the range of Aa/U at which experiments have proved possible, the displacement has been found to vary little with Aa/U.

On the first difficulty, Hall points out (**H**, p. 146) that, if a sphere were used as a Pitot tube, then, provided the round opening in the front of the sphere were large enough* to include the displaced position of the stagnation point (see (30) above for our estimate of this position), the pressure measured would be close to the stagnation pressure on the dividing streamline. He goes on to argue that this 'spherical Pitot tube' would in many ways be equivalent to an ordinary Pitot tube of somewhat smaller diameter. This is well borne out by the subsequent work of Livesey (1956), who tested a Pitot tube with a hemispherical nose and a ratio $d_i/d_e = 0{\cdot}5$ of the internal and external diameters. The studies of § 4 indicate that the shape of the front of the tube, which agrees with that of the sphere, is more important than that of the rear (which extends as a long cylinder instead of being terminated), since the trailing vorticity is unimportant, and so this tube is probably a lot closer than conventional ones are to the ideal 'spherical Pitot tube'. (The main difference between the secondary flows in the two cases will be further discussed at the end of this section.) The measured displacement was $0{\cdot}10d_e$, as compared with $0{\cdot}16d_e$ obtained by Livesey

* He notes that a small hole would in any case be inacceptable because of the resulting sensitivity to yaw.

(as well as by other workers) on tubes of the form used by Young & Maas (1936). This observed difference is in agreement with Hall's suggestion.

Hall then goes on to meet the second difficulty by saying that the displacement function which we have to explain is not a constant, but rather is of the form

$$\frac{\delta}{a} = C \operatorname{sgn}\left(\frac{Aa}{U}\right), \tag{41}$$

where $\operatorname{sgn} x$ is $+1$ when $x > 0$ and is -1 when $x < 0$. (For the displacement is in the direction y increasing if $A > 0$ and in the direction y decreasing if $A < 0$.) In (41) the observed value of the constant C is about 0·2 (since the external diameter d_e of the 'spherical Pitot tube' just discussed is $2a$). Now, the discontinuous behaviour (41) is not plausible on theoretical grounds, and it is much more reasonable to suppose that the true behaviour is something like

$$\frac{\delta}{a} = C \tanh\left(\lambda \frac{Aa}{U}\right), \tag{42}$$

say (where any odd function tending to 1 at $+\infty$ could really be substituted for the tanh), but that λ is large enough so that the tanh takes the value $+1$ or -1 for all Aa/U at which accurate measurement is possible (and indeed for small Aa/U the effect measured is so small that the experimental points become intolerably scattered). If (42) were correct the limit of $U\delta/Aa^2$ as $Aa/U \to 0$ (found in § 3 to be 0·9) would be $C\lambda$, so that values $C = 0{\cdot}2$ and $\lambda = 4{\cdot}5$ would be consistent with the results of this paper.

To test these ideas one should find the next term in the expansion of δ/a in powers of Aa/U, which according to (42) would be

$$\frac{\delta}{a} = C\left\{\lambda \frac{Aa}{U} - \frac{1}{3}\lambda^3 \left(\frac{Aa}{U}\right)^3 + \ldots\right\} = 0{\cdot}9\frac{Aa}{U} - 6{\cdot}1\left(\frac{Aa}{U}\right)^3 + \ldots \tag{43}$$

with the values of C and λ just suggested. The evaluation of the next term therefore requires a study of both the tertiary and quartary flows (that is, both the square and cube terms in the expansion of the velocity field in powers of Aa/U).

Accordingly, Hall evaluates these, with the aid of his basic approximation (described above at the beginning of § 4). After fairly lengthy calculations, he obtains for the displacement effect

$$\frac{\delta}{a} = 1{\cdot}24\frac{Aa}{U} - 1{\cdot}18\left(\frac{Aa}{U}\right)^3 + \ldots. \tag{44}$$

The coefficient of $(Aa/U)^3$ is of the right sign, but is five times too small for agreement with (43). Now, some of this disagreement may be due to selection of the special function tanh in (42); if one took simply a cubic levelled off constant beyond its maximum, the coefficient 6·1 in (43) would be replaced by 2·7. However, even with this form, the effect of quartary upwash on the displacement is still definitely underestimated by Hall's theory.

This is hardly surprising. In the first place, the cumulative error in applying Hall's approximation three times to obtain successively the secondary, tertiary and quartary flows must greatly exceed the error in applying it once only, which already produced an error in the displacement of the order of 30%. Again, Hall's approximation gives erroneously only the divergence, not the curl, of the secondary flow field in the plane of symmetry; but both divergence and curl will be erroneous in the case of the tertiary and quartary flows.

A more clear-cut argument, which also appears to explain the sign of the error in Hall's calculation, is provided by some discussion in **D**, § 3, which is greatly amplified and extended in a forthcoming paper (Lighthill 1957 b).

There it is shown that far upstream the tertiary flow tends to zero more slowly than the secondary flow, and that similarly the quartary flow is greater in magnitude than the tertiary (not, in fact, tending to zero at all). Hall's equations give a smaller order of magnitude for the quartary upwash, at least far upstream, than that indicated by the exact theory. This indicates that the true reduction of the displacement by the quartary flow may be greater than Hall predicts.

Actually, the successive-approximation sequence is not uniformly valid for large r. In this region the disturbances can best be treated as a small perturbation of the exact parallel flow, which leads to their expression as a Hankel transform of suitable solutions of the steady inviscid case of the Orr–Sommerfeld equation for the shear layer.

The solution so obtained can be represented as the sum of secondary, tertiary, quartary flows and so on, with downwashes of orders s^{-2}, s^{-1}, 1, respectively, only for moderately small s. For larger s the downwash behaves as a more complicated function, which ultimately falls off like s^{-4} as $s \to \infty$. The solution of this paper which assumes that the falling-off is like s^{-2} all the way therefore overestimates the displacement. It can be shown (Lighthill 1957 b) that an approximate form of the correction to δ due to departure from the secondary flow for large s is

$$\frac{a^3}{8U}\int_{-\infty}^{\infty}\frac{1}{y}\frac{dA}{dy}\,dy, \tag{45}$$

where the shear A has been supposed a function of y (measured from an origin at the centre of the sphere). This term of order a^3 is negative, and is intermediate in order between the secondary flow term $0{\cdot}9Aa^2/U$ of this paper and Hall's quartary-flow term of order a^4. This again gives a reason for the fairly rapid turning over of the graph of δ/a against Aa^2/U.

There is one more difficulty in the comparison between the theory for the sphere and the experimental results, which can be investigated effectively in the light of the theory just described. It can take two forms.

First, if an actual sphere were used as a Pitot tube, the fact that the sphere experiences drag, and has the wake associated with its drag, modifies the upstream character of the primary irrotational flow, especially far from the body. There it is asymptotic not to a 'doublet' motion but to that due to a source of strength $D/\rho U$, where D is the drag. Accordingly the upstream behaviour assumed in our solution is incorrect. The weight of this objection should not, however, be overestimated. For a sphere of radius a the doublet strength is $2\pi U a^3$, while the source strength even under the high-drag conditions obtaining with laminar separation (with, say, $C_D = \frac{1}{2}$) is only $\frac{1}{4}\pi U a^2$. Hence the source produces a velocity reduction upstream of the body greater than that due to the doublet for $r > 16a$ only, so that it could hardly be expected to be important.

Secondly, for a real Pitot-tube form, the upstream behaviour of the primary irrotational flow will again be asymptotically that of a source, this time of strength $\pi U a^2$, where a is the external radius. Obviously it is more questionable in this case whether the results calculated for the sphere are applicable, even though the Pitot tube may have a hemispherical nose, because of the larger source strength. In both cases the problem is made more serious by the result (**D**, (29)), which gives

$$D(s) \sim \frac{Am}{4\pi U s}, \tag{46}$$

where m is the source strength. If the downwash really satisfied (46), then the integral (31) for the displacement would be logarithmically infinite.

However, it is shown in the forthcoming paper already mentioned that, when the solution for large s is replaced by a uniformly valid approximation, a finite value for δ emerges. It is shown that the displacement so obtained is the same as that deduced from the secondary flow alone if, in the latter, the integration be carried only to the value $s = s_c$, described as the secondary-flow cut-off. In other words the displacement is the same as if the secondary flow penetrated only a distance s_c ahead of the source. The distance s_c is of the order of the width of the shear layer; an approximation to it is given by

$$\log s_c = \frac{1}{2A(0)} \int_{-\infty}^{\infty} \left(-\frac{dA}{dy} \operatorname{sgn} y \right) \log|y| \, dy, \tag{47}$$

which makes s_c in a sense a 'geometric mean' of the difference in y-coordinate between points in the shear layer and the source.

These considerations lead to a term

$$\frac{Am}{4\pi U} \log \frac{s_c}{a} = \frac{a^2}{8U} \int_{-\infty}^{\infty} \left(-\frac{dA}{dy} \operatorname{sgn} y \right) \log \frac{|y|}{a} \, dy \tag{48}$$

in the expansion of δ in powers of a, where in (48) the value of m appropriate to a solid Pitot tube of external diameter $d_e = 2a$ has been inserted. The term (48) of order $a \log a$ in δ/a is in addition to the previously found term of order a. The logarithmic term certainly is a non-negligible addition in this case; when the integral is as much as $2A$, it would increase the coefficient of Aa/U by about a quarter. In the other case, when only the source resulting from the wake drag is present, the effect is not very important. In both cases, however, the effect is not a complete change of order of magnitude but rather a mere quantitative increase.

To sum up, we may state as the main conclusion of this section that the original Young & Maas suggestion of a discontinuous functional dependence (41) of δ/a on Aa/U is without theoretical support. All the theories agree in predicting a continuous functional dependence, which one can only assume (in the light of experimental results) becomes slowly-varying for values of $|Aa/U|$ in excess of a fairly modest limit, an assumption which in the light of the full discussion is seen to be not inconsistent with the theories.

We may ask, in conclusion, whether the work of this paper gives any explanation of two results which were noted as especially interesting in the introduction: the great reduction of the displacement effect with supersonic flow (Johannesen & Mair 1952) or with sharp-lipped tubes (Livesey 1956). Certainly no precise answer is possible in either case, but it is probably relevant to both facts that, as shown in § 4, the image vorticity is the main contributor to the diplacement effect.

Thus, in supersonic flow about a Pitot tube, the extent of regions of vorticity whose image vorticity could affect the downwash on the dividing streamline is much reduced—partly because the region of subsonic flow is cut off by a shock wave a short distance upstream of the orifice, and partly because the flow on streamlines as far away from the axis as $\rho_0/a = 1$ (which were found in the calculations of § 2 to make substantial contributions to the downwash) becomes supersonic very soon, and ceases to have a domain of influence which includes the dividing streamline.

Similarly, one may imagine that the ideal Pitot tube with internal and external diameters equal (to which Livesey's sharp-lipped tube is a good approximation) produces very little displacement because image vorticity in such a tube is reduced, both in magnitude and in effectiveness for producing displacement. For, first, the *undisturbed* vorticity distribution $(0, 0, -A)$ has no image vorticity at all in this case. In other words, the 'first part' of the secondary flow, as defined in **D**, § 3 and in § 2 above (and which for a sphere has the potential (3)), vanishes, since the uniform shearing motion by itself possesses no velocity component normal to the surface.

As to the image system of the vorticity change $\boldsymbol{\omega}_1$, one may note that this will include a uniform vorticity $(0, 0, +A)$ from some point of the tube downstream inside the tube; for in this region all motion, and hence also all vorticity, is doubtless absent. However, the effect of this vorticity trapped inside the tube is negligible outside it; even in the corresponding two-dimensional case it falls off exponentially, by a factor of $e^{\pi} = 23$ in one diameter.

Thus, we are led finally to ask about the effect of such *disturbance* vorticity as is placed similarly to that which produced the displacement in the case of a sphere. As in that case, the effect of the trailing vorticity will be cancelled out by the effect of its images; and the vorticity upstream of the Pitot tube may be expected to make its effect largely through its image vorticity. However, since there is no room inside the surface in the case of a shape like this, the image vorticity must be rather remote—in mathematical language, it is on another sheet of a Riemann surface (of three dimensions). Accordingly, its image effect upstream may be expected to be less than in the case of a more full-bodied shape.

Again, we have not explained the results by the theory, but merely shown that the theory is not inconsistent with them. Clearly, much more work will be required before theories of shear flow become fully satisfactory and widely applicable. However, the attempt to achieve this end seems to be worth making.

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